Transportation Problem

A special class of linear programming problem is Transportation Problem, where the objective is to minimize the cost of distributing a product from a number of sources (e.g. factories) to a number of destinations (e.g. warehouses) while satisfying both the supply limits حدود التجهيز and the demand requirement متطلبات الحاجة. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given rout is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity سلعة, including among others, inventory control, employment scheduling, and personnel assignment. The origin of transportation models dates back to 1941. The objective is to minimize the cost of transportation.

Transportation models

We begin our discussion of transportation problems by formulating a linear programming model of the following situation.

Example:
Powerco has three electric power plants (**sources**) that supply the needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant(1) 35 million; plant(2) 50 million; plant(3) 40 million (see Table 1). The peak power demands in these cities, which occur at the same time are as follows (in kwh): city(1) 45 million; city(2) 20 million; city(3) 30 million; city(4) 30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

Solution: To formulate Powerco's problem as an LP, we begin by defining a variable for each decision that Powerco must make. Because Powerco must determine how much power is sent from each plant to each city, we define (for i =1, 2, 3 and j = 1, 2, 3, 4)

x_{ij} = number of (million) kwh produced at plant i and sent to city j
In terms of these variables, the total cost of supplying the peak power demands to the 4 cities may be written as

$$8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} \quad (Cost\ of\ shipping\ power\ from\ plant\ 1)$$
$$+9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \quad (Cost\ of\ shipping\ power\ from\ plant\ 2)$$
$$+14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \quad (Cost\ of\ shipping\ power\ from\ plant\ 3)$$

Powerco faces two types of constraints. First, the total power supplied by each plant cannot exceed the plant's capacity. For example, the total amount of power sent from plant

Shipping Costs, Supply, and Demand for Powerco

	To				
From	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand (million kwh)	45	20	30	30	

The LP formulation of Powerco's problem contains the following three **supply constraints**:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \quad (plant\ 1\ supply\ constriant)$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50 \quad (plant\ 2\ supply\ constriant)$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \quad (plant\ 3\ supply\ constriant)$$

A constraint that ensures that a location receives its demand is a **demand constraint.** Powerco must satisfy the following four demand constraints:

$$x_{11} + x_{21} + x_{31} \geq 45 \quad (City\ 1\ demand\ constriant)$$
$$x_{12} + x_{22} + x_{32} \geq 20 \quad (City\ 2\ demand\ constriant)$$
$$x_{13} + x_{23} + x_{33} \geq 30 \quad (City\ 3\ demand\ constriant)$$
$$x_{14} + x_{24} + x_{34} \geq 30 \quad (City\ 4\ demand\ constriant)$$

Because all the $x_{ij}'s$ must be nonnegative, we add the sign restrictions $x_{ij} \geq 0$ (i =1, 2,3; j= 1, 2, 3, 4).

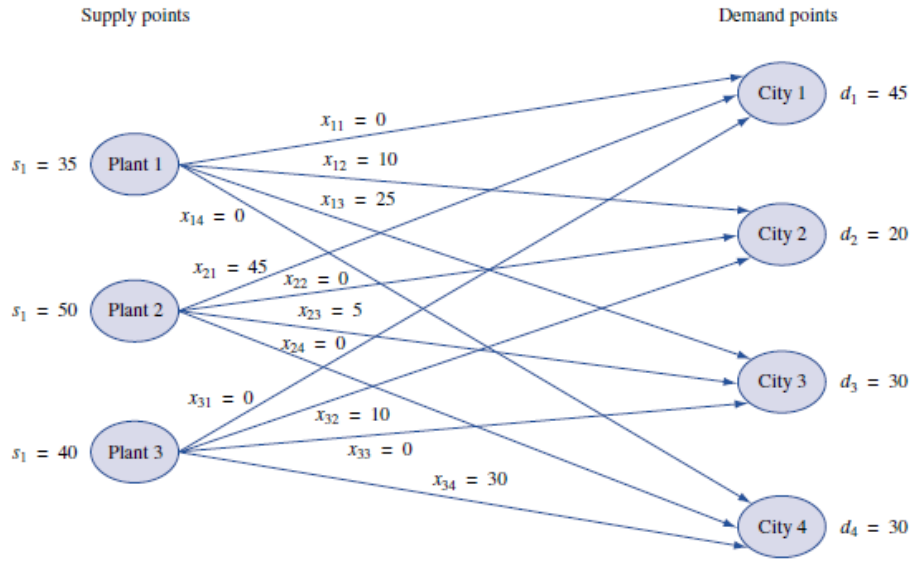
Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following LP formulation of Powerco's problem:

$$\min z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \quad +14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}$$

$$s.t \quad x_{11}+x_{12}+x_{13}+x_{14} \le 35$$
$$x_{21}+x_{22}+x_{23}+x_{24} \le 50 \quad (supply\ constriant)$$
$$x_{31}+x_{32}+x_{33}+x_{34} \le 40$$

$$x_{11}+x_{21}+x_{31} \ge 45 \quad (demand\ constriant)$$
$$x_{12}+x_{22}+x_{32} \ge 20$$
$$x_{13}+x_{23}+x_{33} \ge 30$$
$$x_{14}+x_{24}+x_{34} \ge 30$$
$$x_{ij} \ge 0 \quad (i = 1,2,3;\ j = 1,2,3,4)$$

the optimal solution to this LP is z =1020, $x_{12}=10$, $x_{13}=25$, $x_{21}=45$, $x_{23}=5$, $x_{32}=10$, $x_{34}=30$. Figure 1 is a graphical representation of the Powerco problem and its optimal solution. The variable x_{ij} is represented by a line, or arc, joining the *i*th supply point (plant *i*) and the *j*th demand point (city *j*).



Graphical Representation of Powerco Problem and Its Optimal Solution

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General formula to the problem:

Suppose that there are **m** sources and **n** destinations.

- Let **(s_i)** be the number of supply units available at source **i,** (i = l, 2, 3, ..., m).
- Let (d_j) be the number of demand units required at destination **j**, (j = 1, 2, 3, ..., n).
- Let **(c_{ij})** represent the unit transportation cost for transporting the units from source **i** to destination **j**.
- Let **(x_{ij})** is the number of units shipped from source **i** to destination **j**.

- The objective is to determine the number of units to be transported from source **i** to destination **j** so that the total transportation cost is minimum.

If

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

then total supply equals total demand, and the problem is said to be a **balanced transportation problem.** The problem may be written as

$$\text{Min} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1,2,\dots,m \qquad (Supply\ constraint)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1,2,\dots,n \qquad (Demand\ constraint)$$

$$x_{ij} \geq 0, \ (i = 1,2,\dots,m; \quad j = 1,2,\dots,n)$$

We can represent any transportation problem like the following table.

					Supply
	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

Transportation Tableau for Powerco example

	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6	10	9	35
		10	25		
Plant 2	9	12	13	7	50
	45		5		
Plant 3	14	9	16	5	40
		10		30	
Demand	45	20	30	30	

z =1020, $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$

Basic Feasible Solution

Transportation problem containing **(n + m)** of **structured constraints** and **(n × m)** of **variables**. A feasible solution to a transportation problem is said to be a **basic feasible solution** if the number basic variables = **(m + n – 1)**

We can find the basic feasible solutions using one of the following ways:

1- North –west corner

2- Least cost

3- Vogel approximation method (penalty method)

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

3- Vogel Approximation Method (VAM)

VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1: Write the differences of the smallest and the second smallest costs in each column below the corresponding column.

Step 2: Write the differences of the smallest and the second smallest costs in each row to the right of the corresponding row.
These individual differences are regarded as a **penalty** for making assignments in the second lowest cost cell in place of the lowest cost cell in each row (or column).

Step 3: Select the row or column for which the penalty is the largest and put the maximum possible amount to the cell with the lowest cost in that particular row or column. If there is more than one largest penalty rows or columns, then select any one of them.

Step 4: Cross (or leave) row or column in which the requirement (or demand) has been satisfied and then construct the reduced matrix. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 5: Continue this process on the reduced till all the allocations for this method is finished.

Example Use the **(VAM)** to find the solution to the following problem:

Destinations / Source	D_1	D_2	D_3	D_4	Supply
S_1	10	2	20	11	15
S_2	12	7	9	20	25
S_3	4	14	16	18	10
Demand	5	15	15	15	

Solution: $\sum s_i = \sum d_j = 50$

Destinations / Source	D_1	D_2	D_3	D_4	Supply			
S_1	10 **0**	2 **15**	20 **0**	11 **0**	15	8	9	
S_2	12 **0**	7 **0**	9 **15**	20 **10**	25	2	2	11
S_3	4 **5**	14 **0**	16 **0**	18 **5**	10	10	2	2
Demand	5	15	15	15				
	6	5	7	7				
		5	7	7				
			7	2				

Basic variables are: …….

Non basic variables are: ……

The total cost for transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

Z= 475

Example: Solve the following transportation problem, using Vogel's approximation method.

	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

	D_1	D_2	D_3	Supply	penalty
O_1	2 5	7	4	5	2
O_2	3	3	1	8	2
O_3	5	4	7	7	1
O_4	1	6	2	14	1
Demand	7	9	18	34	
penalty	1	1	1		

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Unbalanced transportation problem

If in a transportation problem where

$$\sum_{i=1}^{i=m} s_i \neq \sum_{j=1}^{j=n} d_j \, .$$

Then the problem is known as unbalanced problem. This problem can be solved by:

This can be done by introducing a (dummy) source or destination. The cost of transporting a unit item from dummy source is taken zero. Similarly, the cost of transporting a unit to the dummy destination is also taken zero.

Example: Transportation model with Dummy Plant

	D_1	D_2	**Supply**
C_1	**80**	**215**	1000
C_2	**100**	**108**	1300
C_3	**102**	**68**	1200
Demand	2300	1400	

$$\sum s_i \neq \sum d_j \rightarrow 3500 \neq 3700$$

	D_1	D_2	Supply
C_1	**80**	**215**	1000
C_2	**100**	**108**	1300
C_3	**102**	**68**	1200
Dummy plant	**0**	**0**	200
Demand	2300	1400	$\sum s_i = \sum d_j = 3700$

Example: Find the initial feasible solution using the VAM to the following transportation problem:

	D_1	D_2	Supply
C_1	**80**	**215**	1000
C_2	**100**	**108**	1500
C_3	**102**	**68**	1200
Demand	1900	1400	

Example: Find the initial feasible solution using the VAM to the following transportation problem:

	Shop A	**Shop B**	**Shop D**	**Shop C**	**Available**
Warehouse I	**3**	**16**	**9**	**2**	40
Warehouse II	**1**	**9**	**3**	**8**	20
Warehouse III	**4**	**5**	**2**	**5**	50
Requirements	25	25	42	8	

Solution: $\sum s_i \neq \sum d_j \rightarrow 110 \neq 100$

	Shop A	**Shop B**	**Shop D**	**Shop C**	**Dummy**	**Available**
Warehouse I	**3**	**16**	**9**	**2**	**0**	**40**
Warehouse II	**1**	**9**	**3**	**8**	**0**	**20**
Warehouse III	**4**	**5**	**2**	**5**	**0**	**50**
Requirement	**25**	**25**	**42**	**8**	**10**	

$$\sum s_i = \sum d_j = 110$$

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	Shop A	Shop B	Shop D	Shop C	Dummy	Available
Warehouse I	3	16	9	2	0	**40**
Warehouse II	1	9	3	8	0	**20**
Warehouse III	4	5	2	5	0	**50**
Requirement	**25**	**25**	**42**	**8**	**10**	

Basic variables are: …….

Non basic variables are: ……

The total cost for transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

Z = ?

The number basic variables = m + n – 1 = 5+3 -1=7

So this solution is initial feasible solution and ?

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H.W: Solve the following Transportation Problem using VAM method

	W1	W2	W3	W4	Available
F1	1	2	1	4	30
F2	3	3	2	1	50
F3	4	2	5	9	20
Requirements	20	40	30	10	

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